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The dual Yoshiara construction gives new extended generalized quadrangles

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Abstract

A Yoshiara family is a set of $q+3$ planes in $\mathrm{PG}(5,q)$, q even, such that for any element of the set the intersection with the remaining $q+2$ elements forms a hyperoval. In 1998 Yoshiara showed that such a family gives rise to an extended generalized quadrangle of order $(q+1, q-1)$. He also constructed such a family $\mathcal{S}(\mathcal{O})$ from a hyperoval $\mathcal{O}$ in $\mathrm{PG}(2,q)$. In 2000 Ng and Wild showed that the dual of a Yoshiara family is also a Yoshiara family. They showed that if $\mathcal{O}$ has o-polynomial a monomial and $\mathcal{O}$ is not regular, then the dual of $\mathcal{S}(\mathcal{O})$ is a new Yoshiara family. This article extends this result and shows that in general the dual of $\mathcal{S}(\mathcal{O})$ is a new Yoshiara family, thus giving new extended generalized quadrangles.

1. Introduction

An *extended generalized quadrangle* of *order* (s,t) is a connected geometry with three types of elements, *points*, *lines* and *blocks* such that the point-residues are generalized quadrangles of order (s,t), the block-residues are isomorphic to the complete graph K_{s+2} on $s+2$ vertices and the line-residues are generalized digons.

The residue of an element x of an extended generalized quadrangle Γ is denoted by $\mathrm{Res}(x)_\Gamma$. We say that Γ is an *extension* of the generalized quadrangle $\mathcal{S}$ if $\mathrm{Res}(x)_\Gamma \cong \mathcal{S}$ for every point x of Γ. Given two extended generalized quadrangles Γ and $\overline{\Gamma}$, a *covering* from $\overline{\Gamma}$ to Γ is an incidence-preserving mapping p from $\overline{\Gamma}$ to Γ inducing an isomorphism from $\mathrm{Res}(x)_{\overline{\Gamma}}$ to $\mathrm{Res}(x)_\Gamma$ for every element x of $\overline{\Gamma}$. The extended generalized quadrangle $\overline{\Gamma}$ is called an *m-fold cover* of Γ if all fibres of p (sets $p^{-1}(x)$, x an element of Γ) have size m. The extended generalized quadrangle Γ is said to be an *m-fold quotient* of $\overline{\Gamma}$.

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In [5], Yoshiara gave a method for constructing extended generalized quadrangles of order $(q+1, q-1)$ from a set $\mathcal{S} = \{\pi_0, \ldots, \pi_{q+2}\}$ of $q+3$ planes in $\mathrm{PG}(5, q)$, q even. The set $\mathcal{S}$ satisfies the following conditions (†):

$$\left.\begin{array}{l}\text{(a) the intersection of two planes } \pi_i, \pi_j \text{ of } \mathcal{S} \text{ is a point for all } i, j \in \{0, \ldots,\\ \quad q+2\}, i \neq j;\\ \text{(b) for each } i \in \{0, \ldots, q+2\}, \text{ the set } \mathcal{O}_i = \{\pi_i \cap \pi_j \mid j \in \{0, \ldots, q+2\} \backslash\\ \quad \{i\}\} \text{ is a hyperoval in } \pi_i; \text{ and}\\ \text{(c) the planes in } \mathcal{S} \text{ span } \mathrm{PG}(5, q).\end{array}\right\} \quad (\dagger)$$

We call a set $\mathcal{S}$ of $q+3$ planes satisfying (†) a *Yoshiara family*. In [5] Yoshiara constructed such a family from a hyperoval $\mathcal{O}$ of $\mathrm{PG}(2, q)$ which we denote $\mathcal{S}(\mathcal{O})$. Thas [4] constructed one from a $(q+1)$-arc $\mathcal{K}$ in $\mathrm{PG}(3, q)$, denoted $\mathcal{S}(\mathcal{K})$.

In [5], given a Yoshiara family $\mathcal{S} = \{\pi_0, \pi_1, \ldots, \pi_{q+2}\}$ in $\mathrm{PG}(5, q)$, Yoshiara constructed an extended generalized quadrangle $\mathcal{Y}_q(\mathcal{S})$ of order $(q+1, q-1)$ in the following way. Embed $\mathrm{PG}(5, q)$ as a hyperplane in $\mathrm{PG}(6, q)$. Points of $\mathcal{Y}_q(\mathcal{S})$ are the three-dimensional subspaces of $\mathrm{PG}(6, q)$ which contain an element of $\mathcal{S}$ but are not contained in $\mathrm{PG}(5, q)$. Lines of $\mathcal{Y}_q(\mathcal{S})$ are the lines of $\mathrm{PG}(6, q)$ which contain one of the points $\pi_i \cap \pi_j, i \neq j$, but are not contained in $\mathrm{PG}(5, q)$. Blocks of $\mathcal{Y}_q(\mathcal{S})$ are the points of $\mathrm{PG}(6, q) \backslash \mathrm{PG}(5, q)$. Incidence is given by symmetrized inclusion. If Σ is a point of $\mathcal{Y}_q(\mathcal{S})$ meeting $\mathrm{PG}(5, q)$ in the plane π_i, $i \in \{0, 1, \ldots, q+2\}$, containing hyperoval $\mathcal{O}_i$ by intersection with the remaining $q+2$ planes, then the residue of $\mathcal{Y}_q(\mathcal{S})$ at Σ is, by definition, equivalent to the dual of the GQ $T_2^*(\mathcal{O})$ (see [5, Lemma 2.1]).

We will denote $\mathcal{Y}_q(\mathcal{S}(\mathcal{O}))$ by $\mathcal{Y}_q(\mathcal{O})$ and $\mathcal{Y}_q(\mathcal{S}(\mathcal{K}))$ by $\mathcal{Y}_q(\mathcal{K})$.

Let $\mathcal{S} = \{\pi_0, \ldots, \pi_{q+2}\}$ be a Yoshiara family, and consider its dual $\mathcal{S}' = \{\pi'_0, \ldots, \pi'_{q+2}\}$ in $\mathrm{PG}(5, q)$. Note that $\mathcal{S}'$ is a set of $q+3$ planes satisfying conditions (†′):

$$\left.\begin{array}{l}\text{(a}'\text{) every pair of planes } \pi'_i, \pi'_j \text{ in } \mathcal{S}' \text{ span a hyperplane;}\\ \text{(b}'\text{) for each } i \in \{0, \ldots, q+2\}, \text{ the set } \mathcal{O}'_i = \{\langle \pi'_i, \pi'_j \rangle \mid j \in \{0, \ldots, q+2\} \backslash\\ \quad \{i\}\} \text{ is a dual hyperoval containing } \pi'_i \text{ (that is, a set of } q+2 \text{ hyperplanes}\\ \quad \text{containing } \pi'_i \text{ such that no 3 have a 3-space in common); and}\\ \text{(c}'\text{) the intersection of the planes in } \mathcal{S}' \text{ is the empty space.}\end{array}\right\} \quad (\dagger')$$

Ng and Wild [3] show that the dual $\mathcal{S}'$ of a Yoshiara family $\mathcal{S}$ satisfies conditions (†) and so is also a Yoshiara family. In [4] Thas observes that $\mathcal{S}(\mathcal{K})$ consists of $q+1$ generators of a fixed $Q^+(5, q)$ and two planes that are polar with respect to the polarity of the $Q^+(5, q)$. Hence $\mathcal{S}(\mathcal{K})$ is self-dual. However, this is not true in general for the Yoshiara construction $\mathcal{S}(\mathcal{O})$. By the fundamental theorem of projective geometry any hyperoval $\mathcal{O}$ in $\mathrm{PG}(2, q)$ is equivalent to one of the form $\{(1, t, f(t)) \mid t \in \mathrm{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$ with $f(0) = 0$ and $f(1) = 1$. Such a function f is called an *o-polynomial*. A hyperoval is called *monomial* if it is equivalent to a hyperoval with o-polynomial of the form x^k, $k \in \mathbb{Z}$.

Ng and Wild [3] show that if $\mathcal{O}$ is monomial and $\mathcal{O}$ is not regular, then the dual of $\mathcal{S}(\mathcal{O})$, denoted $\mathcal{S}'(\mathcal{O})$, is not isomorphic to $\mathcal{S}(\mathcal{O})$. Thus $\mathcal{S}'(\mathcal{O})$ gives rise to a new family of extended generalized quadrangles which we denote $\mathcal{Y}'_q(\mathcal{O})$.

In this article we show that the dual of $\mathcal{S}(\mathcal{O})$ is isomorphic to $\mathcal{S}(\overline{\mathcal{O}})$ if and only if $\mathcal{O}$ is projectively equivalent to $\overline{\mathcal{O}}$ and $\mathcal{O}$ is a regular hyperoval. Thus all the Yoshiara families constructed by Ng and Wild using the dual are new. This gives rise to many new extended generalized quadrangles.

2. The Yoshiara construction $\mathcal{S}(\mathcal{O})$

We begin by describing the construction of a set $\mathcal{S}(\mathcal{O})$ of $q+3$ planes in $\mathrm{PG}(5,q)$, q even satisfying (†) due to Yoshiara [5]. Let $\mathcal{O}$ be a hyperoval of $\mathrm{PG}(2,q)$, q even and let $\mathcal{O}^*$ be its dual. Let ϕ be the bijection from the points of $\mathrm{PG}(2,q)$ onto the Veronese surface $\mathcal{V}_2^4$ in $\mathrm{PG}(5,q)$ (see [2] for a detailed description of $\mathcal{V}_2^4$ and its properties):

$$\begin{aligned}\phi\colon\quad & \mathrm{PG}(2,q) \longrightarrow \mathrm{PG}(5,q)\\ & (x_0,x_1,x_2) \longmapsto (x_0^2,x_1^2,x_2^2,x_0x_1,x_1x_2,x_0x_2).\end{aligned}$$

The $q+2$ lines $\ell_0,\ldots,\ell_{q+1}$ of $\mathcal{O}^*$ are mapped by ϕ onto $q+2$ conics $\mathcal{C}_{\ell_0},\ldots,\mathcal{C}_{\ell_{q+1}}$ of $\mathcal{V}_2^4$. A plane of $\mathrm{PG}(5,q)$ meeting $\mathcal{V}_2^4$ in a conic is called a *conic plane*. Each conic $\mathcal{C}_{\ell_i}$ has a nucleus N_i that lies in the nucleus plane of $\mathcal{V}_2^4$. The set $\{N_0,\ldots,N_{q+1}\}$ forms a hyperoval in the nucleus plane. Let π_{ℓ_i} be the conic plane containing the hyperoval $\mathcal{O}_i = \mathcal{C}_{\ell_i} \cup \{N_i\}$, $i = 0,\ldots,q+1$ and let π be the nucleus plane of $\mathcal{V}_2^4$ containing the hyperoval $\mathcal{O}_{q+2} = \{N_0,\ldots,N_{q+1}\}$. The set of planes $\mathcal{S}(\mathcal{O}) = \{\pi_{\ell_0},\ldots,\pi_{\ell_{q+1}},\pi\}$ satisfies conditions (†). The hyperovals $\mathcal{O}_0,\ldots,\mathcal{O}_{q+1}$ are regular, while the hyperoval $\mathcal{O}_{q+2}$ is projectively equivalent to $\mathcal{O}$.

Note that the Thas construction $\mathcal{S}(\mathcal{K})$ [4] has two regular hyperovals and $q+1$ translation hyperovals. The two constructions $\mathcal{S}(\mathcal{O})$ and $\mathcal{S}(\mathcal{K})$ are isomorphic if and only if $\mathcal{O}$ is regular and $\mathcal{K}$ is a twisted cubic [4].

3. The main results

Ng and Wild showed that if $\mathcal{O}$ is a monomial hyperoval, then all the hyperovals in the dual of $\mathcal{S}(\mathcal{O})$ are projectively equivalent to $\mathcal{O}$. Thus if $\mathcal{O}$ monomial and is not a regular hyperoval, then the Yoshiara family is new. We show that for any hyperoval $\mathcal{O}$, all the hyperovals in $\mathcal{S}'(\mathcal{O})$ are projectively equivalent to $\mathcal{O}$. Thus if $\mathcal{O}$ is a hyperoval with o-polynomial not a monomial, then the Yoshiara family $\mathcal{S}'(\mathcal{O})$ is not isomorphic to any of the known Yoshiara families. This answers an open question in [3].

We use the following results about the Veronese surface $\mathcal{V}_2^4$.

Lemma 3.1 (See Chapter 25 of [2]). *Suppose ϕ is the bijection from the set of points of $\mathrm{PG}(2,q)$ to $\mathcal{V}_2^4$ as in Section 2. Then*

1. *A conic plane of $\mathcal{V}_2^4$ meets the nucleus plane of $\mathcal{V}_2^4$ in exactly one point.*
2. *Two conic planes of $\mathcal{V}_2^4$ meet in exactly one point.*
3. *Under ϕ a line ℓ of $\mathrm{PG}(2,q)$, considered as a quadric of $\mathrm{PG}(2,q)$, determines a unique hyperplane of $\mathrm{PG}(5,q)$ meeting $\mathcal{V}_2^4$ in exactly the conic corresponding to ℓ considered as a set of points.*

4. *Under ϕ two distinct lines ℓ and m of* PG$(2, q)$, *considered as a quadric of* PG$(2, q)$, *determine a unique hyperplane of* PG$(5, q)$ *meeting $\mathcal{V}_2^4$ in exactly the two conics corresponding to ℓ and m considered as sets of points.*

Theorem 3.2. *Let $\mathcal{O}$ be a hyperoval of* PG$(2, q)$ *and $\mathcal{S}'(\mathcal{O}) = \{\pi'_0, \ldots, \pi'_{q+2}\}$ the corresponding dual Yoshiara family. Then for each $i = 0, 1, \ldots, q+2$ the hyperoval $\mathcal{O}'_i = \{\pi_i \cap \pi_j \mid j = 0, 1, \ldots, q+2, i \neq j\}$ is projectively equivalent to $\mathcal{O}$.*

Proof. We begin by investigating some representations of PG$(2, q)$ in PG$(5, q)$ using the Veronese surface $\mathcal{V}_2^4$. Let ϕ be the Veronesean map from the points of PG$(2, q)$ to $\mathcal{V}_2^4$ as in Section 2 and let π be the nucleus plane of $\mathcal{V}_2^4$. If ℓ is a line of PG$(2, q)$, then ϕ maps the points of ℓ to a conic $\mathcal{C}_\ell$ of $\mathcal{V}_2^4$ that lies in a conic plane π_ℓ of $\mathcal{V}_2^4$.

Define the map T_π from PG$(2, q)$ to PG$(5, q)$ by:

$$T_\pi(P) = \langle \pi, \phi(P) \rangle \qquad \text{for } P \text{ a point of PG}(2, q),$$
$$T_\pi(\ell) = \langle \pi, \pi_\ell \rangle \qquad \text{for } \ell \text{ a line of PG}(2, q).$$

Let $P \in$ PG$(2, q)$ and suppose that there exists a point $Q \in$ PG$(2, q)$, $P \neq Q$, such that $\phi(Q) \in T_\pi(P)$. The line PQ of PG$(2, q)$ corresponds to a conic $\mathcal{C}_{PQ}$, through $\phi(P)$ and $\phi(Q)$, contained in a conic plane π_{PQ}. The nucleus of $\mathcal{C}_{PQ}$ is in the nucleus plane π, thus $\pi_{PQ} \in \langle \pi, \phi(P) \rangle$, and so π_{PQ} meets π in a line, a contradiction. Hence no such point Q exists, that is, $\langle \pi, \phi(P) \rangle \cap \mathcal{V}_2^4 = \phi(P)$. We also note that the hyperplane $\langle \pi, \pi_\ell \rangle$ contains exactly one conic plane, namely π_ℓ. Since T_π also preserves incidence, T_π defines an isomorphism from PG$(2, q)$ to the quotient space PG$(5, q)/\pi$.

Now let ℓ be a line of PG$(2, q)$ and let AG$(2, q)^\ell$ be the affine plane constructed from PG$(2, q)$ by removing ℓ. Define the map T_ℓ from AG$(2, q)^\ell$ to PG$(5, q)$ by:

$$T_\ell(P) = \langle \pi_\ell, \phi(P) \rangle \qquad \text{for } P \text{ a point of AG}(2, q)^\ell,$$
$$T_\ell(m) = \langle \pi_\ell, \pi_m \rangle \qquad \text{for } m \text{ a line of AG}(2, q)^\ell.$$

Let P be a point of AG$(2, q)^\ell$ and suppose that there exists a point Q of AG$(2, q)^\ell$ such that $\phi(Q) \in T_\ell(P)$. Let $R = \ell \cap PQ$ in PG$(2, q)$, then $\phi(P), \phi(Q), \phi(R) \in \langle \pi_\ell, \phi(P) \rangle$ and so the conic plane π_{PQ} is in $\langle \pi_\ell, \phi(P) \rangle$. This gives a contradiction as π_{PQ} and π_ℓ meet in a point. Thus $\langle \pi_\ell, \phi(P) \rangle$ meets $\mathcal{V}_2^4$ in the point $\phi(P)$ and the conic $\mathcal{C}_\ell$ corresponding to ℓ. Further, the hyperplane $\langle \pi_\ell, \pi_m \rangle$ contains precisely two conic planes, namely π_ℓ and π_m. As T_ℓ also preserves incidence, T_ℓ defines an isomorphism from AG$(2, q)^\ell$ to $(\text{PG}(5, q)/\pi_\ell) \backslash (\langle \pi, \pi_\ell \rangle / \pi_\ell)$. We can extend this to an isomorphism from PG$(2, q)$ to PG$(5, q)/\pi_\ell$ by defining $T_\ell(\ell) = \langle \pi, \pi_\ell \rangle$ and $T_\ell(P) = \langle \pi, \phi(P) \rangle$ for points $P \in \ell$.

Now let $\mathcal{O}$ be a hyperoval of PG$(2, q)$ and let $\mathcal{O}^* = \{\ell_0, \ldots, \ell_{q+1}\}$ be its dual. Let $\pi_{\ell_0}, \ldots, \pi_{\ell_{q+1}}$ be the corresponding conic planes of $\mathcal{V}_2^4$ and so $\mathcal{S}(\mathcal{O}) = \{\pi_{\ell_0}, \ldots, \pi_{\ell_{q+1}}, \pi\}$ is the corresponding Yoshiara family. Note that

$$T_\pi(\mathcal{O}^*) = \{\langle \pi, \pi_{\ell_i} \rangle \mid i = 0, 1, \ldots, q+1\}, \qquad \text{and} \tag{1}$$
$$T_{\ell_i}(\mathcal{O}^*) = \{\langle \pi_{\ell_i}, \pi_{\ell_j} \rangle \mid j \in \{0, \ldots, q+1\} \backslash \{i\}\} \cup \{\langle \pi, \pi_{\ell_i} \rangle\}. \tag{2}$$

Let $\mathcal{S}'(\mathcal{O}) = \{\pi'_{\ell_0}, \ldots, \pi'_{\ell_{q+1}}, \pi'\}$ be the dual of $\mathcal{S}(\mathcal{O})$ in PG$(5, q)$. As $\mathcal{S}'(\mathcal{O})$ is also a Yoshiara family, we have a hyperoval in each of these planes. The hyperoval in π' is

$\mathcal{O}'_{q+2} = \{\pi' \cap \pi'_{\ell_j} \mid j \in \{0, \ldots, q+1\}\}$, and the hyperoval in π'_{ℓ_i} is $\mathcal{O}'_i = \{\pi'_{\ell_i} \cap \pi'_{\ell_j} \mid j \in \{0, \ldots, q+1\} \backslash \{i\}\} \cup \{\pi'_{\ell_i} \cap \pi'\}$ for $i = 0, \ldots, q+1$.

The dual of $\mathcal{O}'_{q+2}$ is the set $\{\langle \pi, \pi_{\ell_j} \rangle \mid j \in \{0, \ldots, q+1\}\}$. By (1), this is $T_\pi(\mathcal{O}^*)$ and as T_π is an isomorphism from PG$(2, q)$ to PG$(5, q)/\pi$, the set is isomorphic to $\mathcal{O}^*$. Since dualizing the plane PG$(5, q)/\pi$ in PG$(5, q)$ gives the plane π', it follows that $\mathcal{O}'_{q+2}$ is equivalent to $\mathcal{O}$. Similarly, for $i \in \{0, \ldots, q+1\}$, if we consider the hyperoval $\mathcal{O}'_i$ in the plane π_{ℓ_i}, its dual is the set $\{\langle \pi_{\ell_i}, \pi_{\ell_j} \rangle \mid j \in \{0, \ldots, q+1\} \backslash \{i\}\} \cup \{\langle \pi_{\ell_i}, \pi \rangle\}$ which is isomorphic to $\mathcal{O}^*$ by (2). Dualizing PG$(5, q)/\pi_{\ell_i}$ in PG$(5, q)$ gives π_{ℓ_i} so $\mathcal{O}'_i$ is isomorphic to $\mathcal{O}$ for $i = 0, \ldots, q+1$. Thus all the hyperovals $\mathcal{O}'_0, \ldots, \mathcal{O}'_{q+2}$ in the dual Yoshiara family $\mathcal{S}'(\mathcal{O})$ are projectively equivalent to $\mathcal{O}$. $\square$

Corollary 3.3. *The extended generalized quadrangle $\mathcal{Y}'_q(\mathcal{O})$ is isomorphic to $\mathcal{Y}_q(\overline{\mathcal{O}})$ if and only if $\mathcal{O}$ is projectively equivalent to $\overline{\mathcal{O}}$ and $\mathcal{O}$ is a regular hyperoval.*

Proof. Suppose that $\mathcal{Y}'_q(\mathcal{O})$ and $\mathcal{Y}_q(\overline{\mathcal{O}})$ are isomorphic. Each point residue of $\mathcal{Y}'_q(\mathcal{O})$ is isomorphic to the dual of $T_2^*(\mathcal{O})$, and the point residues of $\mathcal{Y}_q(\overline{\mathcal{O}})$ are isomorphic to either $T_2^*(\overline{\mathcal{O}})$ or $T_2^*(\mathcal{R})$, where $\mathcal{R}$ is a regular hyperoval. Thus it follows that $T_2^*(\mathcal{O}) \cong T_2^*(\overline{\mathcal{O}})$ and $T_2^*(\mathcal{O}) \cong T_2^*(\mathcal{R})$. Hence by Bichara et al. [1] we have that $\mathcal{O}$ is projectively equivalent to $\overline{\mathcal{O}}$ and $\mathcal{O}$ is a regular hyperoval.

Next, if $\mathcal{O}$ and $\overline{\mathcal{O}}$ are both regular hyperovals, then by Thas [4] we have that $\mathcal{Y}_q(\overline{\mathcal{O}}) \cong \mathcal{Y}_q(\mathcal{K})$, where $\mathcal{K}$ is a twisted cubic of PG$(3, q)$. Since $\mathcal{S}(\mathcal{K})$ is self-dual it follows that $\mathcal{Y}'_q(\mathcal{O}) \cong \mathcal{Y}_q(\overline{\mathcal{O}})$. $\square$

Corollary 3.4. *The extended generalized quadrangle $\mathcal{Y}'_q(\mathcal{O})$ is an extension of the dual of $T_2^*(\mathcal{O})$.*

4. Quotients of $\mathcal{Y}'_q(\mathcal{O})$

By [4] if there is a point of PG$(5, q)$ not contained in the span of any two elements of a Yoshiara family, then the extended generalized quadrangle arising from this Yoshiara family admits a q-fold quotient (essentially by projection onto a hyperplane). Thas also showed that both $\mathcal{Y}_q(\mathcal{O})$ and $\mathcal{Y}_q(\mathcal{K})$ admit q-fold quotients. In [3] Ng and Wild prove that $\mathcal{Y}'_q(\mathcal{O})$ admits a q-fold quotient in the case where $\mathcal{O}$ is a monomial hyperoval. Generalizing their proof we prove the general case.

Theorem 4.1. *The extended generalized quadrangle $\mathcal{Y}'_q(\mathcal{O})$ admits a q-fold quotient.*

Proof. Consider the Yoshiara family $\mathcal{S}(\mathcal{O}) = \{\pi_0, \ldots, \pi_{q+2}\}$ and its dual $\mathcal{S}'(\mathcal{O}) = \{\pi'_0, \ldots, \pi'_{q+2}\}$. There exists a point of PG$(5, q)$ not contained in any $\langle \pi'_i, \pi'_j \rangle$, $i, j = 0, \ldots, q+2, i \neq j$, if and only if there is a hyperplane of PG$(5, q)$ containing no point of the form $\pi_i \cap \pi_j$, $i, j = 0, \ldots, q+2, i \neq j$. Now suppose that $\mathcal{O}^*$ is the dual of the hyperoval $\mathcal{O}$ in PG$(2, q)$ and P is a point of PG$(2, q)$ incident with no line of $\mathcal{O}^*$. Let $\mathcal{Q}$ be a quadric in PG$(2, q)$ containing just the point P. Then under ϕ, the bijection of points of PG$(2, q)$ onto $\mathcal{V}_2^4$, the equation of $\mathcal{Q}$ becomes the equation of a hyperplane of PG$(5, q)$ meeting $\mathcal{V}_2^4$ in just $\pi(P)$, with $\pi(P) \notin \pi_i$, $i = 0, \ldots, q+2$. $\square$

Addendum

The authors thank the anonymous referee for the following remarks, the details of which have been omitted for the sake of brevity.

Remark 4.2. Theorem 3.2 may be verified by explicit calculation based on coordinates. By using these explicit coordinates Theorem 1 of [6] may be used to show that in the case where $\mathcal{O}$ is a monomial hyperoval and $q > 4$ that $\mathcal{Y}'_q(\mathcal{O})$ is simply connected.

Remark 4.3. In the case where $\mathcal{O}$ is a Lunelli-Sce hyperoval, which admits an automorphism group transitive on its points, the geometry $\mathcal{Y}'_q(\mathcal{O})$ is not flag-transitive.

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